# MAT 210 Discrete Convolution (Cyclic and Infinite)

**Goal:** Given a data vector $\underline{f}$ we want to modify $\underline{f}$ to change frequency properties.

---

**Example:** Suppose $\underline{f}$ is a 5 second recording of someone talking. The human voice is mostly in the 50 – 4,000 Hz frequency range. Removing frequencies outside this range should leave the voice okay while removing hums and buzzes.

---

**MATLAB:**

```
>> filter = (50*5 < 1:length(voice)) .*
            (1:length(voice) < 4000*5);
>> cleanVoice = ifft(fft(voice).*filter', 'symmetric')
```

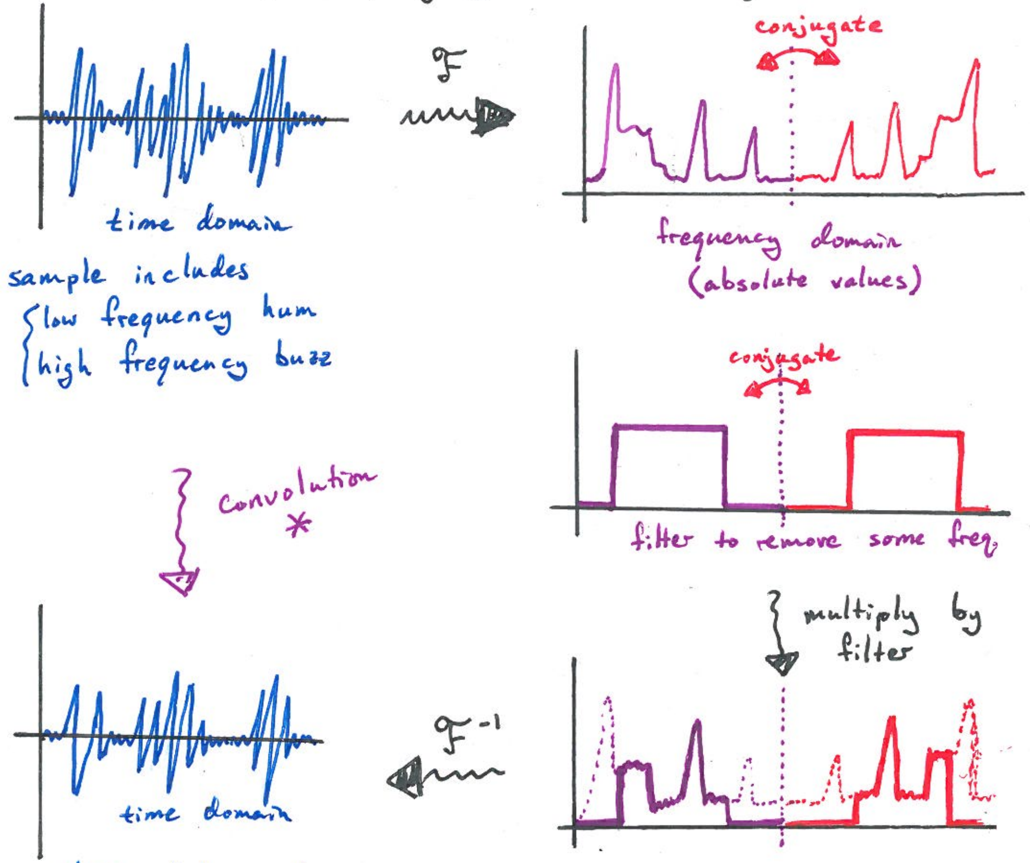
**Problems:** ① Not possible in real time

- cannot begin without entire recording

② Requires lots of computation

**Solution:** Convolutions!

Convolutions modify the Fourier coefficients of $\underline{f}$ by changing $\underline{f}$ directly.




clean data no longer has hum or buzz.

The continuous convolution formula is (basically) the same for $\mathcal{F}$ as for $\mathcal{L}$ in MAT 219

(MAT 219)

**Definition:** $(f*g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$

**Property:** $\mathcal{L}\{f*g\} = \mathcal{L}\{f\}\cdot\mathcal{L}\{g\}$

- For Laplace transforms, $f$ & $g$ were defined on $[0,\infty)$ (or $=0$ for $t<0$)
- For Fourier transforms, $f$ & $g$ are defined on $[0, 2\pi)$ (periodic for other $t$)

↳ This changes the formula for $f*g$ slightly: $g(t-\tau)=0$ if $\tau>t$ ‖ in Laplace world, but not in Fourier world.

**Def:** The continuous convolution of $f(t)$ & $g(t)$ is

$$(f*g)(t) = \int_0^{2\pi} f(\tau)\, g(t-\tau)\, d\tau$$

**Property:** $\mathcal{F}_k\{f*g\} = 2\pi\, \mathcal{F}_k\{f\}\cdot\mathcal{F}_k\{g\}$



**Note:** Some people will either define Fourier Transform differently ($\mathcal{F}_k = \int_0^{2\pi} f\, e^{-ikt}\, dt$) or the convolution differently ($f*g = \frac{1}{2\pi}\int_0^{2\pi} f(\tau)\, g(t-\tau)\, d\tau$) to remove the $2\pi$ in the convolution theorem

A quick proof of Convolution Thm:

$$\mathcal{F}_k\{f*g\} = \frac{1}{2\pi}\int_0^{2\pi} (f*g)\, e^{-ikt}\, dt$$

$$= \frac{1}{2\pi}\int_{t=0}^{t=2\pi}\left(\int_{\tau=0}^{\tau=2\pi} f(\tau)\, g(t-\tau)\, d\tau\right) e^{-ikt}\, dt$$

(change order of integ.) $$= \frac{1}{2\pi}\int_{\tau=0}^{\tau=2\pi}\int_{t=0}^{t=2\pi} f(\tau)\, g(t-\tau)\, e^{-ikt}\, dt\, d\tau$$

$$= \frac{1}{2\pi}\int_{\tau=0}^{\tau=2\pi} f(\tau)\left(\int_{t=0}^{t=2\pi} g(t-\tau)\, e^{-ikt}\, dt\right) d\tau$$

(Subst. $t = z+\tau$, $dt = dz$ in inner int.) $$= \frac{1}{2\pi}\int_{\tau=0}^{\tau=2\pi} f(\tau)\left(\int_{z=-\tau}^{z=2\pi-\tau} g(z)\, e^{-ik(z+\tau)}\, dz\right) d\tau$$

$$= \frac{1}{2\pi}\left(\int_0^{2\pi} f(\tau)\, e^{-ik\tau}\, d\tau\right)\left(\int_0^{2\pi} g(z)\, e^{-ikz}\, dz\right)$$

$$= 2\pi\, \mathcal{F}_k\{f\}\cdot\mathcal{F}_k\{g\}$$

Note that $g(z)e^{-ikz}$ is periodic so $\int_{-\tau}^{2\pi-\tau} g(z)\, e^{-ikz}\, dz = \int_0^{2\pi} g(z)\, e^{-ikz}\, dz$

∎

**Side note:** We can compute convolution in "real-time":

Suppose $g(t)$ is a signal being measured

→ every second we get more of $g(t)$

and $f(t)$ is a filter which someone computed

$$\mathcal{F}_k\{f\} = \begin{cases} 0 & \text{if } k \text{ is a frequency we wish to remove from } g \\ 1 & \text{if } k \text{ is a frequency we wish to keep in } g \end{cases}$$

To compute $(f*g)(t) = \int_0^{2\pi} f(\tau)\, g(t-\tau)\, d\tau$

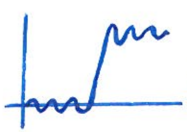
you only need to know $g(t-\tau)$ for $0 \le \tau \le 2\pi$

- $g(t-0) = $ (current value of signal)
- $g(t-\tau) = $ (past values of signal)

→ Unlike Fourier transform, it is not necessary to know all values of $g(t)$ before beginning computation!

---

In practice, filters $f(t)$ don't have $\mathcal{F}_k\{f\} = \begin{cases} 0 \\ 1 \end{cases}$ because that would make $f$ very complicated and $f*g$ would be annoying to compute.

Usually $\mathcal{F}\{f\}$ is more like  "equiripple" filter

## Discretize!

| $t$ | $f$ | $g$ |
|---|---|---|
| $t_0 = 0$ | $f_0 = f(t_0)$ | $g_0 = g(t_0)$ |
| $t_1 = \frac{2\pi}{N}$ | $f_1 = f(t_1)$ | $g_1 = g(t_1)$ |
| $\vdots$ | $\vdots$ | $\vdots$ |
| $t_n = \frac{2\pi}{N} n$ | | $g_n = g(t_n)$ |
| $t_m = \frac{2\pi}{N} m$ | $f_m = f(t_m)$ | |

$$\boxed{\begin{aligned} t_{n-m} &= \frac{2\pi}{N}(n-m) \\ &= \frac{2\pi}{N} n - \frac{2\pi}{N} m = t_n - t_m \end{aligned}}$$

$$\int f(\tau)\, g(t-\tau)\, d\tau$$

$$\sum_{m=0}^{N-1} f(t_m)\, \underbrace{g(t_n - t_m)}_{= g(t_{n-m})} \cancel{\frac{2\pi}{N}}$$

$$\sum_{m=0}^{N-1} f_m \cdot g_{n-m} \cancel{\frac{2\pi}{N}}$$

ignore

## Discrete Convolution

$$(f*g)_n = \sum_{m=0}^{N-1} f_m \cdot g_{n-m} = \sum_{i+j=n} f_i \cdot g_j$$

(**Note:** This may involve $g_{-1}$ $g_{-2}$ $g_{-3}$ etc...)

There are **two** different versions of this formula, depending on how you treat $g_{-1}, g_{-2}$, etc.

- (Infinite) convolution $(f*g)$ ← $g_{-1}, g_{-2}$, etc are 0
- Cyclic convolution $(f \circledast g)$ ← $g_{-1}, g_{-2}$, etc come from periodicity

# Infinite Convolution $f * g$

$$(f*g)_n = \sum_{\substack{i+j=n \\ i,j\geq 0}}' f_i \cdot g_j$$

There is a simple and beautiful way to calculate this... but first we will use force.

EX: $(\underset{f_0}{1}, \underset{f_1}{2}, \underset{f_2}{4}) * (\underset{g_0}{2}, \underset{g_1}{1}, \underset{g_2}{3})$

$$(f*g)_0 = f_0 g_0 = 1 \cdot 2 = 2 \qquad (i+j=0 \Rightarrow \underline{i=0} \ \& \ \underline{j=0})$$

$$(f*g)_1 = f_0 g_1 + f_1 g_0 = 1\cdot 1 + 2 \cdot 2 = 5 \qquad \left(i+j=1 \Rightarrow \underline{i=0} \ \& \ \underline{j=1} \text{ or } \underline{i=1} \ \& \ \underline{j=0}\right)$$

$$(f*g)_2 = f_0 g_2 + f_1 g_1 + f_2 g_0 = 1\cdot 3 + 2\cdot 1 + 4 \cdot 2 = 13$$

$$(f*g)_3 = f_1 g_2 + f_2 g_1 = 2\cdot 3 + 4 \cdot 1 = 10$$

$$(f*g)_4 = f_2 g_2 = 4 \cdot 3 = 12$$

$$(f*g) = (2, 5, 13, 10, 12)$$

**Note:** $f*g$ is a different length than $f$ & $g$

→ In fact $f*g$ is defined even if $f$ & $g$ have different lengths.

---

$f*g$ is called "infinite convolution" because it is "stabilized"

$$(1,2,4,0,0,0)*(2,1,3,0,0,0) = (2,5,13,10,12,0,0,0,0,0,0)$$

→ In general

$$(f, 0, 0, \ldots, 0) * (g, 0, 0, \ldots 0) = (f*g, 0, 0, \ldots 0)$$

---

→ **Better way to compute:**

Step #1 Insert a variable to keep track of index

EX $(1, 2, 4) \rightsquigarrow 1 + 2x + 4x^2$

$(2, 1, 3) \rightsquigarrow 2 + x + 3x^2$

index $i=0$ $i=1$ $i=2$ $\rightsquigarrow$ $x^0$ $x^1$ $x^2$ power of $x$

Convolution = Polynomial Multiplication

$(1,2,4)*(2,1,3)$ $\qquad$ $(2, 5, 13, 10, 12)$

$$(1+2x+4x^2)(2+x+3x^2) = 2 + 5x + 13x^2 + 10x^3 + 12x^4$$

**Step #2** Polynomial mult. can be organized like "long mult." of numbers (without carrying)

**EX** $213 \times 421$ ⇝

$$\begin{array}{rrrrr}
 & & 10^2 & 10^1 & 10^0 \\
 & & 2 & 1 & 3 \\
\times & & 4 & 2 & 1 \\
\hline
 & & 2 & 1 & 3 \\
 & 4 & 2 & 6 & \\
+\,8 & 4 & 2 & & \\
\hline
8 & 9 & 8 & 7 & 3
\end{array}$$

$\|$

$89{,}873$

$(2+x+3x^2)\cdot(4+2x+x^3)$ ⇝

$$\begin{array}{rrrrrl}
 & & x^2 & x & x^0 & \\
 & & 3 & 1 & 2 & \leftarrow (3x^2+x+2) \\
 & & 1 & 2 & 4 & \leftarrow (x^2+2x+4) \\
\times & & & & & \\
\hline
 & & 12 & 4 & 8 & \\
 & 6 & 2 & 4 & & \\
+\,3 & 1 & 2 & & & \\
\hline
3 & 7 & 16 & 8 & 8 &
\end{array}$$

$\|$

$8+8x+16x^2+7x^3+3x^4$ ⇜ $3\ \ 7\ \ 16\ \ 8\ \ 8$

(Note: Do not carry. This is $x^2$ position. $16x^2 \neq x^3+6x^2$)

This even works with negative coefficients

$(-2+10x)(5-2x+x^2)$ ⇝

$$\begin{array}{rrrr}
 & 1 & -2 & 5 \\
 & & 10 & -2 \\
\times & & & \\
\hline
 & -2 & 4 & -10 \\
10 & -20 & 50 & \\
\hline
10 & -22 & 54 & -10
\end{array}$$

$\|$

$-10+54x-22x^2+10x^3$ ⇜ $10\ \ -22\ \ 54\ \ -10$

**Result**: Convolution = Multiplication (without carrying)

**EX**: $(1,2,4) * (2,1,3)$

$$\begin{array}{rrrrr}
 & & 4 & 2 & 1 \\
\times & & 3 & 1 & 2 \\
\hline
 & & 8 & 4 & 2 \\
 & 4 & 2 & 1 & \\
+\,12 & 6 & 3 & & \\
\hline
12 & 10 & 13 & 5 & 2
\end{array}$$

$\boxed{\text{Ans}: (2, 5, 13, 10, 12)}$

**EX**: $(1,-1,2) * (3,4,-2)$

$$\begin{array}{rrrrr}
 & & 2 & -1 & 1 \\
\times & & -2 & 4 & 3 \\
\hline
 & & 6 & -3 & 3 \\
 & 8 & -4 & 4 & \\
+\,-4 & 2 & -2 & & \\
\hline
-4 & 10 & 0 & 1 & 3
\end{array}$$

$\boxed{\text{Ans}: (3, 1, 0, 10, -4)}$

**EX**: $(2,0,-1,3) * (4,5)$

$$\begin{array}{rrrrr}
 & 3 & -1 & 0 & 2 \\
\times & & & 5 & 4 \\
\hline
 & 12 & -4 & 0 & 8 \\
+\,15 & -5 & 0 & 10 & \\
\hline
15 & 7 & -4 & 10 & 8
\end{array}$$

$\boxed{\text{Ans}: (8, 10, -4, 7, 15)}$

**Note:** Computing $f * g$ involves two reversals

(1) Reverse $f$ & $g$ when converting to a product

$$(1,2,3) * (4,5,6) \rightsquigarrow \begin{array}{r} 3\ 2\ 1 \\ \times\ 6\ 5\ 4 \\ \hline \end{array}$$

(2) Reverse result of product to get ans.

$$\begin{array}{rrrrr} & & 3 & 2 & 1 \\ & \times & 6 & 5 & 4 \\ \hline 18 & 27 & 22 & 13 & 4 \end{array} \rightsquigarrow (4, 13, 22, 27, 18)$$

For convolution $f * g$ these two reversals cancel

$$\begin{array}{rrrrr} & & 1 & 2 & 3 \\ & \times & 4 & 5 & 6 \\ \hline 4 & 13 & 22 & 27 & 18 \end{array}$$

Same answer if you do no reverses...

For cyclic convolution, they will not cancel.

BEWARE!

Reversing is important for $f \circledast g$ !

## Cyclic Convolution $f \circledast g$

**Note:** Cyclic convolution $f \circledast g$ is only defined if $\text{length}(f) = \text{length}(g)$.

also $\text{length}(f \circledast g) = \text{length}(f) = \text{length}(g)$

For cyclic convolution $g_{-m}$ is determined by periodicity

$g_{N-2} = g_{-2}$

$g_{N-1} = g_{-1}$

g repeats

$$\underline{g} = \begin{bmatrix} g_0 \\ g_1 \\ \vdots \\ g_{N-1} \end{bmatrix}$$

g repeats

$g_0 = g_N$

$g_1 = g_{N+1}$

$$\rightsquigarrow g_{-m} = g_{N-m}$$

Convolution formula becomes

$$(f \circledast g)_n = \sum_{\substack{i+j \equiv n \\ (\text{mod } N)}} f_i \cdot g_j$$

**Recall:** $i+j \equiv n \pmod N$ means $\frac{i+j}{N}$ has remainder $n$

$1+2 \equiv 0 \pmod 3$, $2+4 \equiv 1 \pmod 5$, etc.

There is a simple and beautiful way to calculate this ... but first we will use force.

**EX:** $(1, 2, 4) \circledast (2, 1, 3)$

$f_0$ " $f_1$ " $f_2$ "  $g_0$ " $g_1$ " $g_2$ "

$(f*g)_3$

$$(f \circledast g)_0 = f_0 \cdot g_0 + \boxed{f_1 \cdot g_2 + f_2 \cdot g_1}$$
$$= 1 \cdot 2 + \boxed{2 \cdot 3 + 4 \cdot 1}$$
$$= 12$$

$i+j \equiv 0 \mod 3$
$i=0$ & $j=0$
or $i=1$ & $j=2$
or $i=2$ & $j=1$

$$(f \circledast g)_1 = f_0 \cdot g_1 + f_1 \cdot g_0 + \boxed{f_2 \cdot g_2}$$
$$= 1 \cdot 1 + 2 \cdot 2 + \boxed{4 \cdot 3}$$
$$= 17$$

$i+j \equiv 1 \mod 3$
$i=0$ & $j=1$
or $i=1$ & $j=0$
or $i=2$ & $j=2$

$(f*g)_4$

$$(f \circledast g)_2 = f_0 \cdot g_2 + f_1 \cdot g_1 + f_2 \cdot g_0$$
$$= 1 \cdot 3 + 2 \cdot 1 + 4 \cdot 2$$
$$= 13$$

**Ans:** $(f \circledast g) = (12, 17, 13)$

**Note:** In the problem above (length 3)

$$(f \circledast g)_0 = (f*g)_0 + (f*g)_3$$
$$(f \circledast g)_1 = (f*g)_1 + (f*g)_4$$

Translate this to long multiplication:

**EX:** $(1, 2, 4) \circledast (2, 1, 3)$

| | | | | |
|---|---|---|---|---|
| | | 4 | 2 | 1 |
| | × | 3 | 1 | 2 |
| | | 8 | 4 | 2 |
| | $\boxed{4}$ | 2 | 1 | $\boxed{4}$ |
| $\boxed{12}$ | $\boxed{6}$ | 3 | $\boxed{12}$ | $\boxed{6}$ |
| | | 13 | 17 | 12 |

**Ans:** $(12, 17, 13)$

**EX:** $(1, 2, 0, -1) \circledast (-1, 3, -2, 1)$

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | -1 | 0 | 2 | 1 |
| | | × | 1 | -2 | 3 | -1 |
| | | | 1 | 0 | -2 | -1 |
| | | $\boxed{-3}$ | 0 | 6 | 3 | $\boxed{-3}$ |
| | $\boxed{2}$ | $\boxed{0}$ | -4 | -2 | $\boxed{2}$ | $\boxed{0}$ |
| $\boxed{-1}$ | $\boxed{0}$ | $\boxed{2}$ | 1 | $\boxed{-1}$ | $\boxed{0}$ | $\boxed{2}$ |
| | | | -2 | 3 | 3 | -2 |

**Ans:** $(-2, 3, 3, -2)$

## Basic Property of Convolution:

$$\mathcal{F}_k(\underline{f} \circledast \underline{g}) = N\, \mathcal{F}_k\{\underline{f}\} \cdot \mathcal{F}_k\{\underline{g}\}$$

and

$$\mathcal{F}_k(\underline{f} \cdot \underline{g}) = \mathcal{F}_k\{\underline{f}\} \circledast \mathcal{F}_k\{\underline{g}\}$$

("Convolution Theorem")

(Convolution in time domain) = (Product in frequency domain) $\cdot N$

(Product in time domain) = (Convolution in frequency domain)

EX: $(1,2,0,-1) \circledast (-1,3,-2,1)$

$= (-2,3,3,-2)$

Verify the convolution theorem.

"Show that

$4 \cdot \mathcal{F}_k(1,2,0,-1) \cdot \mathcal{F}_k(-1,3,-2,1)$

$= \mathcal{F}_k(-2,3,3,-2)$"

**k=0**

$\mathcal{F}_0(1,2,0,-1) = \frac{1}{4}(1+2+0-1) = \frac{1}{2}$

$\mathcal{F}_0(-1,3,-2,1) = \frac{1}{4}(-1+3-2+1) = \frac{1}{4}$

$\mathcal{F}_0(-2,3,3,-2) = \frac{1}{4}(-2+3+3-2) = \frac{1}{2}$

$4 \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{2}$ ok.

**k=1**

$\mathcal{F}_1(1,2,0,-1) = \frac{1}{4}(1-2i-0-i) = \frac{1-3i}{4}$

$\mathcal{F}_1(-1,3,-2,1) = \frac{1}{4}(-1-3i+2+i) = \frac{1-2i}{4}$

$\mathcal{F}_1(-2,3,3,-2) = \frac{1}{4}(-2-3i-3-2i) = \frac{-5-5i}{4}$

$4 \cdot \frac{1-3i}{4} \cdot \frac{1-2i}{4} = \frac{(1-6)+(-3-2)i}{4} = \frac{-5-5i}{4}$ ok

**k=2**

$\mathcal{F}_2(1,2,0,-1) = \frac{1}{4}(1-2+0+1) = 0$

$\mathcal{F}_2(-1,3,-2,1) = \frac{1}{4}(-1-3-2-1) = -\frac{7}{4}$

$\mathcal{F}_2(-2,3,3,-2) = \frac{1}{4}(-2-3+3+2) = 0$

$4 \cdot 0 \cdot (-\frac{7}{4}) = 0$ ok

Do not need to check k=3 because it is conjugate of k=1